PERTH MODERN SCHOOL Exceptional schooling. Exceptional students. Independent Public School	Year 12 Specialist TEST 3 Monday 1 July 2019 TIME: 45 minutes working Classpads allowed One page of notes 42 marks 6 Questions

Name: MARKING KEY

Teacher: ____________________________

Note: All part questions worth more than 2 marks require working to obtain full marks.

Q1 (3 & 3 = 6 marks)

a) Solve for the following system of linear equations.

$$x+2y-z=3$$
$$2x+3y+2z=-1$$
$$3x+7y-2z=6$$

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 2 & 3 & 2 & -1 \\ 3 & 7 & -2 & 6 \end{array}\right]$$

✓ shows working (Gaussian elimination)

✓ solves for one variable

✓ solves for the other two

(No marks if no working shown.)

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -4 & 7 \\ 0 & -1 & -1 & 3 \end{bmatrix} \begin{array}{l} \\ 2R_1-R_2 \\ 3R_1-R_3 \end{array}$$

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -4 & 7 \\ 0 & 0 & -5 & 10 \end{bmatrix} \begin{array}{l} \\ \\ R_2'+R_3' \end{array}$$

$z=-2$

$y-4(-2)=7$

$y=-1$

$x+2(-1)-(-2)=3$

$x=3$

soln $(3,-1,-2)$

Q1 - continued

b) Determine the values of m & p such that $\begin{aligned} x+2y-z&=3 \\ 2x+3y+2z&=m \\ 3x+py-2z&=6 \end{aligned}$ such that the system has

(i) Infinite solutions
(ii) No solutions

$$\left[\begin{array}{ccc:c} 1 & 2 & -1 & 3 \\ 2 & 3 & 2 & m \\ 3 & p & -2 & 6 \end{array}\right]$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & 3 \\ 0 & 1 & -4 & 6-m \\ 0 & 6-p & -1 & 3 \end{array}\right] \begin{array}{l} \\ 2R_1-R_2 \\ 3R_1-R_3 \end{array}$$

$$\left[\begin{array}{cccc} 1 & 2 & -1 & 3 \\ 0 & 1 & -4 & 6-m \\ 0 & 1-4(6-p) & 0 & 6-m-12 \end{array}\right] R_2'-4R_3'$$

$1-4(6-p) = -23+4p$ $\qquad$ $6-m-12 = -6-m$

✓ obtaining a line with two zeros

✓ states values for infinite

✓ states values for no soln

i) Infinite $\quad -23+4p=0$ AND $-6-m=0$

$p=\frac{23}{4}$ AND $m=-6$

ii) No solns $\quad p=\frac{23}{4}$ AND $m \neq -6$

/3

Q2 (2 & 3 = 5 marks)

If $r = \begin{pmatrix} t^2-1 \\ 5-t \end{pmatrix}$, determine

a) $\left|\frac{dr}{dt}\right|$

$\dot{r} = \begin{pmatrix} 2t \\ -1 \end{pmatrix}$ $|\dot{r}| = \sqrt{4t^2+1}$

✓ $\dot{r}$
✓ answer

b) $\frac{d|r|}{dt}$

$|r| = \sqrt{(t^2-1)^2 + (5-t)^2}$ ✓

$\frac{d|r|}{dt} = \frac{1}{2}\left[(t^2-1)^2+(5-t)^2\right]^{-\frac{1}{2}}\left(2(t^2-1)2t + 2(5-t)(-1)\right)$ ✓ ✓

(Note – two marks for answer only)

Q3 (2, 3 & 3 = 8 marks)

An object is initially at the origin with initial speed of $\begin{pmatrix} 3 \\ 7 \end{pmatrix} m/s$ and an acceleration given by

$a = \begin{pmatrix} 3e^t \\ 2-\sin t \end{pmatrix} m/s^2$ at time t seconds.

Obtain an expression for the:

a) Velocity at time t.

✓ integrated
✓ solves for c

$v = \begin{pmatrix} 3e^t \\ 2t+\cos t \end{pmatrix} + c$ $\begin{pmatrix} 3 \\ 7 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + c$ $c = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

$= \begin{pmatrix} 3e^t \\ 2t+\cos t+6 \end{pmatrix}$

b) Position vector r at time t.

$r = \begin{pmatrix} 3e^t \\ t^2+\sin t+6t \end{pmatrix} + c$ ✓ $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \end{pmatrix} + c$ $c = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$ ✓

$r = \begin{pmatrix} 3e^t-3 \\ t^2+\sin t+6t \end{pmatrix}$ ✓

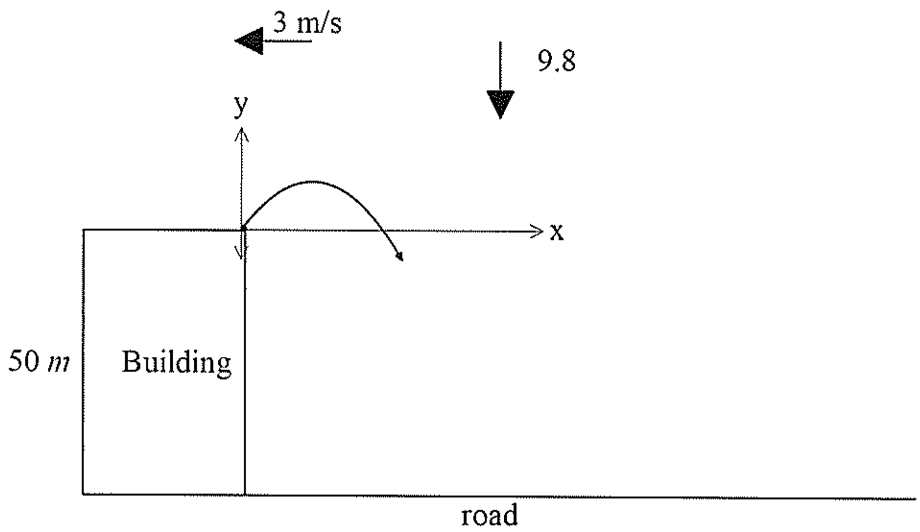
c) Is the velocity ever perpendicular to the acceleration? Explain and if necessary solve for t values (if any).

$a \cdot v = 9e^{2t} + (2-\sin t)(2t+\cos t) > 0$

always greater than zero
∴ will never be perpendicular

✓ uses dot product
✓ gives exp for $a \cdot v$ in terms of t
✓ states that dot product > 0

Q4 (3, 3, 3 & 3 = 12 marks)
Consider a cannon ball that is projected from the top of a building with speed V at an angle θ to the surface of the roof. There is a constant cross wind of 3 metres per second acting against the ball and the acceleration due to gravity is $9.8m/s^2$ down as shown in the diagram below. (Note- let the origin be at the top of the building on the edge)



a) Given that the acceleration is given by $\ddot{r} = \begin{pmatrix} 0 \\ -9.8 \end{pmatrix} m/s^2$ show using vector calculus that the velocity $\dot{r} = \begin{pmatrix} V\cos\theta - 3 \\ V\sin\theta - 9.8t \end{pmatrix} m/s$.

$\dot{\underset{\sim}{r}} = \begin{pmatrix} 0 \\ -9.8t \end{pmatrix} + \underset{\sim}{c}$

✓ integrates
✓ solves for $\underset{\sim}{c}$
✓ states $\underset{\sim}{v}$

$\dot{\underset{\sim}{r}}(0) = \begin{pmatrix} V\cos\theta - 3 \\ V\sin\theta \end{pmatrix}$

$\begin{pmatrix} V\cos\theta - 3 \\ V\sin\theta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \underset{\sim}{c}$

$\dot{\underset{\sim}{r}} = \begin{pmatrix} V\cos\theta - 3 \\ V\sin\theta - 9.8t \end{pmatrix}$

b) Determine the cartesian equation of the path of the cannon ball in terms of V & θ. Show your working.

$\underset{\sim}{r} = \begin{bmatrix} (V\cos\theta - 3)t \\ Vt\sin\theta - 4.9t^2 \end{bmatrix} + \underset{\sim}{c}$

$\underset{\sim}{r}(0) = \underset{\sim}{0} \quad \therefore \underset{\sim}{c} = \underset{\sim}{0}$

✓ states $\underset{\sim}{r}(0) = \underset{\sim}{0}$ or $\underset{\sim}{c} = \underset{\sim}{0}$

$x = (V\cos\theta - 3)t \qquad t = \dfrac{x}{V\cos\theta - 3}$

✓ obtains exp for time in terms of x

$y = Vt\sin\theta - 4.9t^2$

✓ obtains cartesian eqn

$y = \dfrac{xV\sin\theta}{V\cos\theta - 3} - \dfrac{4.9x^2}{(V\cos\theta - 3)^2}$

(−1 if origin changed)
Question states origin at top of building

Q4 continued

c) Given that a point on the cartesian path has been measured as $(7.4, 1.1)$ metres and the initial speed V of the ball from the cannon is 12 m/s, determine the initial angle θ of the ball when projected into the air.

$$1.1 = \frac{7.4(12)\sin\theta}{12\cos\theta - 3} - \frac{4.9(7.4)^2}{(12\cos\theta - 3)^2} \qquad 0 < \theta < \frac{\pi}{2}$$

$$\theta = 30.1^\circ \text{ or } 54.7^\circ$$
$$(0.524^R) \quad (0.954^R)$$

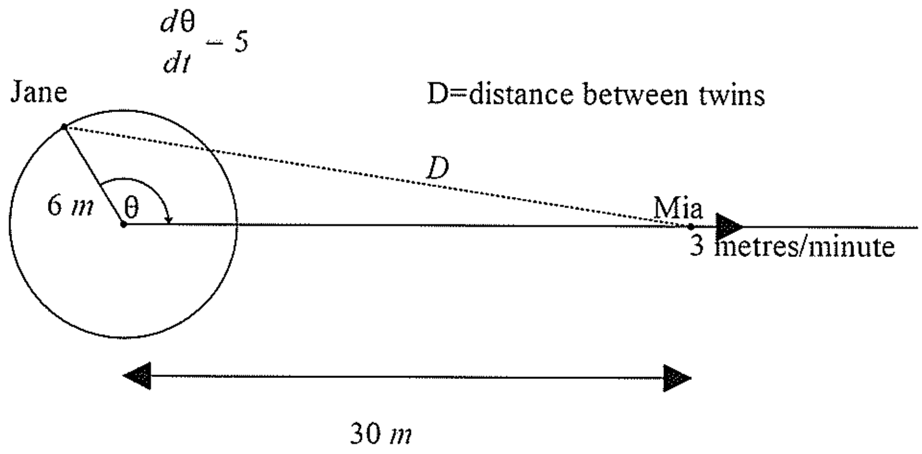
✓ subs known x & y values into cartesian
✓ solves for θ
✓ states 2 solutions (acute)

d) If $V = 25\,m/s$ and $\theta = 45$ and a cross wind of 3 m/s as in the diagram on last page, determine how far from the foot of the building that the cannon ball lands on the road.

$$-50 = 25t\sin 45^\circ - 4.9t^2$$
$$t = -1.86,\ 5.47 \text{ sec}$$
$$t \geq 0 \quad \therefore t = 5.47 \text{ secs}$$
$$x = (25\cos 45^\circ - 3)\,5.47$$
$$= 80.286 \text{ m}$$

✓ sets up eqn for t
✓ subs t value into x
✓ states distance
(No need for units)

Q5 (2 & 5 = 7 marks)
Consider two rides at a circus, one is a merry go round and the other is a train on a straight line. Two twins decide to each try one of the two rides, Jane sits on the merry go round with a constant angular speed of $\frac{d\theta}{dt} = 5$ radians/minute moving in a clockwise direction and radius 6 metres and Mia sits on a train moving at 3 metres/minute away from the merry go round. See the diagram below.



a) Determine the distance between Jane and Mia when $\theta = \frac{2\pi}{3}$ and the train is 30 metres from the centre of the merry go round.

$$D^2 = 6^2 + 30^2 - 2(6)(30)\cos\frac{2\pi}{3}$$
$$D = 33.41m$$

✓ uses cosine rule
✓ states distance (No need for units)

b) Determine the time rate of change of this distance at the point defined in (a) above. (metres/minute)

$$D^2 = 6^2 + x^2 - 2(6)(x)\cos\theta \quad ✓$$

$$2D\dot{D} = \underbrace{2x\dot{x} - 12\dot{x}\cos\theta}_{✓} + 12x\sin\theta\,\dot{\theta} \quad ✓$$

$$\dot{\theta} = -5 \quad ✓$$

$$2(33.41)\dot{D} = 2(30)(3) - 12\cos\tfrac{2\pi}{3}(3) + 12(30)\sin\tfrac{2\pi}{3}(-5)$$

$$\dot{D} = -20.37 \text{ m/min} \quad ✓$$

Q6 (4 marks)

Given that $x \& y$ are functions of t, show that $\dfrac{d^2y}{dx^2} = \dfrac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^3}$

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\dot{y}}{\dot{x}} \quad \checkmark$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} \quad \checkmark$$

$$= \frac{\dfrac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^2}}{\dot{x}} \quad \checkmark$$

$$= \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3} \quad \checkmark$$

$f(t)$